

The Elusive Asymptotic Behavior of the High-Temperature Expansion of the Hierarchical Ising Model

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We present a differential formulation of the recursion formula of the hierarchical model which provides a recursive method of calculation for the high-temperature expansion. We calculate the first 30 coefficients of the high-temperature expansion of the magnetic susceptibility of the Ising hierarchical model with 12 significant digits. We study the departure from the approximation which consists in identifying the coefficients with the values they would take if a $[0, 1]$ Padé approximant were exact. We show that, when the order in the high-temperature expansion increases, the departure from this approximation grows more slowly than for nearest neighbor models. As a consequence, the value of the critical exponent γ estimated using Padé approximants converges very slowly and the estimations using 30 coefficients have errors larger than 0.05. A (presumably much) larger number of coefficients is necessary to obtain the critical exponents with a precision comparable to the precision obtained for nearest neighbor models with fewer coefficients. We also discuss the possibility of constructing models where a $[0, 1]$ Padé approximant would be exact.

KEY WORDS: Renormalization group; critical exponents; hierarchical models; high-temperature expansion; Ising models; epsilon expansion; Padé approximants.

1. MOTIVATIONS, MAIN RESULTS, AND NOTATIONS

The hierarchical model⁽¹⁾ is a model for which the renormalization group transformation⁽²⁾ reduces to a simple recursion formula. This recursion formula is a simple integral equation which has been studied in great detail.⁽³⁾ More recently, interesting results concerning the analyticity and the location of the complex zeros of the probability distribution associated with the

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infrared fixed point have been found.⁽⁴⁾ The hierarchical model has a free parameter ε which is used in the ε -expansion. By adjusting this parameter, one can use the hierarchical model as an approximation⁽¹⁻⁵⁾ to nearest neighbor models in various dimensions. In the case $D = 3$, the hierarchical approximation yields values of the critical exponents which agree within a few percent with the best estimates for nearest neighbor models.

A difficult but interesting question is how to improve the hierarchical approximation while keeping control of the complexity of the renormalization group transformation. This question can be answered for Gaussian models using group-theoretic methods.⁽⁶⁾ However, the extension of these methods to interacting models is not straightforward because many more terms can be generated perturbatively. The main problem consists in identifying the most important perturbations which should be added to the hierarchical model in order to obtain a critical behavior more similar to nearest neighbor models. A possible indicator which could be used to achieve this goal is to calculate the effect of a perturbation on the high-temperature coefficients. Before evaluating the effects of these perturbations, it is necessary to develop efficient methods to calculate the high-temperature expansion of the (unperturbed) hierarchical model. Recently, analytical formulas for the first four coefficients of the magnetic susceptibility were calculated for an Ising measure.⁽⁷⁾ The method used relies on high-temperature graphs. Due to the nonlocality of the model, the terms appearing in the coefficients rapidly proliferate and require elaborate symbolic methods to be handled.

In this article, we present a recursive method which allows an efficient calculation of the high-temperature expansion of the hierarchical model. This method combines the renormalization group method with the high-temperature expansion, but does not rely on any graphical analysis. It is a simple differential version of the recursion relation (with an arbitrary rescaling of the spin variable). It essentially solves the problem of the high-temperature expansion provided that we keep the numerical stability and the volume effects under control. This method is explained in Section 2, where we also calculate the first 30 coefficients of the high-temperature expansion of the magnetic susceptibility of the hierarchical Ising model in a large (2^{60} sites) but finite volume. Checking with analytical results and comparing different implementations of the differential recursion formula, we claim that for $\varepsilon = 1$, the method is numerically stable as far as 12 significant digits are concerned, and that within this precision, the first 30 coefficients have reached their infinite-volume limit when 2^{60} sites are reached. We also briefly discuss the possible use of the differential formula to obtain analytical results concerning the high-temperature expansion.

In Section 3, we study the departure from the approximation which

consists in identifying the coefficients with the values they would take if the $[0, 1]$ Padé approximant for the logarithmic derivative of the susceptibility were exact. The resulting approximate relation among the coefficients was suggested by a numerical study⁽⁸⁾ where we calculated (without any approximation) the magnetic susceptibility of the hierarchical Ising model with up to 2^{18} sites. In this study, we found that the numerical data can be fitted very precisely with a simple power law of the form $(1 - \beta/\beta_0)^{-g}$ in the *whole* high-temperature region, i.e., for $\beta \in [0, \beta_c)$. The values of g obtained from the numerical data or from the $[0, 1]$ approximant differ significantly from the best estimates of the values of the critical exponent γ calculated^(3, 9) with the ε -expansion. In the case of the numerical data, the most plausible explanation provided by a renormalization group analysis⁽⁸⁾ is that this discrepancy is due to the limited size of the lattice used in these calculations. As we shall recall in the following section, if the lattice has 2^n sites, the recursion formula can only be used n times, and $n = 17$ does not seem to be enough to get rid of the irrelevant components of the measure. On the other hand, the fact that a $[0, 1]$ approximant gives a poor estimate is not surprising. What is surprising is how slowly the departure from this approximation grows when the order in the high-temperature expansion increases.

In Section 4, we use Padé approximants to estimate γ and β_c . We first discuss how the results depend on the volume. We show that when the volume is not too large—less or much less than 2^{20} sites for the orders considered here—the approximants can resolve a pair of complex conjugate roots approaching the real axis in the complex temperature plane. However, for larger volume, two real roots appear with—in a large majority of cases—only one root stabilizing near β_c . This stabilization toward the infinite-volume seems to occur with a precision comparable to the precision of the coefficients. We checked this statement for the six approximants which can be calculated from the exact⁽⁷⁾ values of the first four coefficients at infinite volume. We concluded that the errors on the estimation of the critical quantities due to finite-volume or roundoff errors were very small compared to the size of the effects discussed below. More precisely, the 12-significant-digits precision on the infinite-volume coefficients is more than sufficient to discuss effects of order 10^{-3} in the critical quantities. The results for the 435 approximants that one can obtain from the coefficients up to order 30 are then presented. The estimate of β_c are slightly too high for low-order approximants, but seem to converge reasonably well (i.e., small errors in the fourth significant digit in most of the cases) when the order becomes large enough. On the other hand, the estimates of γ are significantly too high for low-order approximants and evolve very slowly toward our best estimates. In order to give an idea,

absolute errors of order 0.07 and 0.11 (respectively) are typical of the large-order estimations for $\varepsilon = 1$ and 0 (respectively).

These large discrepancies can be understood in terms of the approximate relations discussed in Section 3. For instance, the values of the critical exponent γ extracted from a $[9, 10]$ Padé is still midway between the crude estimate given by the $[0, 1]$ approximant (e.g., 1.52 for $\varepsilon = 1$) and the exact value (1.30, in the same case). This point is discussed at length in Section 4 for $\varepsilon = 1$ and 0. From these results, it appears that a (presumably much) large number of coefficients is necessary to obtain the critical exponents of the hierarchical model with precision comparable to these obtained⁽¹⁰⁾ for nearest neighbor model (i.e., a precision of 0.001 or better).

We also discuss (Section 5) the possibility of constructing models where a $[0, 1]$ Padé approximant would be exact. We first compare the approximate relations for the hierarchical model and the nearest neighbor models when the dimensional parameter $D = 4 - \varepsilon$ is varied. We show that the corrections to the Gaussian result (for which the approximation is exact) appear only at second order in the $1/D$ expansion⁽¹²⁾ for the hierarchical model, while it appears in first order for nearest neighbor models. Unfortunately, the use of this expansion is obscured by a proliferation of zeros and poles appearing in the expression of the coefficients when D is promoted to a complex variable.⁽⁷⁾ We also mention the fact that supersymmetric models⁽¹³⁾ might be used as a guide to achieve this goal.

In the conclusions, we give various estimates of the number of coefficients necessary to calculate γ with a reasonable precision. We discuss briefly the feasibility of the project and additional problems which could be considered in this context. We also compare with other recursive methods⁽¹¹⁾ used to calculate the high-temperature expansion of the $3D$ Ising model. This raises the following dilemma: on one hand, we have a much simpler and efficient method of calculation; on the other hand, we need many more coefficients to obtain comparable information.

In order to give a self-contained presentation, we briefly recall our conventions concerning the hierarchical Ising model and its free parameter. Hierarchical models⁽¹⁾ are specified by a nonlocal Hamiltonian bilinear in the spin variables and a local measure of integration. In the following, we consider exclusively the case of an Ising measure where the spin variables take only the values ± 1 . The models considered here have 2^n sites. In the following, n will always be used in this sense. We label the sites with n indices x_n, \dots, x_1 , each index being 0 or 1. The energy of a spin configuration reads

$$H = -\frac{1}{2} \sum_{l=1}^n \left(\frac{c}{4}\right)^l \sum_{x_n, \dots, x_{l+1}} \left(\sum_{x_l, \dots, x_1} \sigma_{(x_n, \dots, x_1)} \right)^2 \quad (1.1)$$

The motivations for this construction and the derivation of the recursion formula are reviewed in ref. 8.

The model has a free parameter c for which we shall use the parametrization

$$c = 2^{1-2/D} \quad (1.2)$$

The parameter of the ε -expansion can be defined as

$$\varepsilon = 4 - D \quad (1.3)$$

When $D \geq 4$, the model has a trivial continuum limit.⁽³⁾ When $D \leq 2$, the model does not have a phase transition at finite temperature.⁽¹⁾ These two rigorous results can be understood heuristically in terms of the self-intersection properties of the random walk associated with H , by noticing that the Hausdorff dimension of this random walk is $2/D$.⁽¹⁴⁾ From the point of view discussed at the beginning of the introduction, the most interesting region is $2 \leq D \leq 4$ (i.e., $1 \leq c \leq 2^{1/2}$); however, the model has a sensible infinite-volume limit for a wider range of the free parameter, namely $0 < D < \infty$ (i.e., $0 < c < 2$). We define the magnetic susceptibility per site as

$$\chi_n(\beta) = \frac{1}{2^n} \left\langle \left(\sum_x \sigma_x \right)^2 \right\rangle_n \quad (1.4)$$

The high-temperature expansion of this quantity reads

$$\chi_b(\beta) = 1 + b_{1,n}\beta + b_{2,n}\beta^2 + \dots \quad (1.5)$$

2. A RECURSIVE METHOD FOR THE CALCULATION OF THE HIGH-TEMPERATURE EXPANSION

Due to the nonlocality of the hierarchical model, the evaluation of the coefficients is rather tedious. We have used three types of techniques to calculate these coefficients. Two of them have been described at length in ref. 7. Up to order 4, analytical calculations can be performed using algebraic methods⁽⁷⁾ and yield expressions where D and n are arbitrary. Substituting numerical values into these formulas provides very precise numerical comparisons with other numerical methods. We checked the validity of the analytical results by introducing a high-temperature expansion in the numerical method used in ref. 8. This numerical method is very reliable but rather slow and does not allow large- n calculations, because the computer time involved grows like 4^n .

In order to perform large- n calculations, we have designed another method of calculation based on the Fourier transform version of the

recursion relation. If we call $\hat{P}_n(k)$ the Fourier transform of the probability of the mean spin for 2^n sites, we obtain the recursion relation

$$\hat{P}_{n+1}(k) = C_{n+1} \exp \left[-\frac{1}{2} \beta \left(\frac{c}{4} \right)^{n+1} \frac{\partial^2}{\partial k^2} \right] (\hat{P}_n(k))^2 \quad (2.1)$$

where $C_n + 1$ is adjusted in order to get $\hat{P}_{n+1}(0) = 1$ and the exponential of the second derivate is defined through the Taylor expansion. For the Ising model considered here, the initial function is

$$\hat{P}_0(k) = \cos(k) \quad (2.2)$$

In this formulation, the magnetic susceptibility reads

$$\chi_n(\beta) = -\frac{1}{2^n} \frac{\partial^2}{\partial k^2} \hat{P}_n(k) \Big|_{k=0} \quad (2.3)$$

More generally, $\hat{P}_n(k)$ generates the average values for the even powers of the mean spin:

$$\hat{P}_n(k) = \sum_{m=0}^{\infty} \frac{(-ik)^{2m}}{2m!} \left\langle \left(\sum_x \sigma_x \right)^{2m} \right\rangle_n \quad (2.4)$$

Equation (2.1) is a differential formulation of the recursion formula. Unlike the original formulation, it does not involve any integrals and it can be reduced to a sequence of purely algebraic operations when expanded in powers of β to a given order. In other words, (2.1) can be seen as a recursive definition of the high-temperature expansion of the average values of the even powers of the mean spin. This simple procedure completely solves the problem of the high-temperature expansion of the hierarchical model, provided that the operations can be performed within a reasonable amount of time and provided that we can reach the infinite-volume limit with an acceptable numerical precision.

The computer time necessary to calculate the m first coefficients grows like $m^2 n$ since we only need to retain the first $2m + 2$ terms in the Taylor expansion of the functions introduced in the recursion formula. This is much better than the 4^n behavior reached by straightforward evaluation⁽⁷⁾ following the integration method of ref. 8. In order to give an idea concerning the absolute time scale, it takes a little less than 2 days, with Mathematica, to iterate 60 times Eq. (2.1) expanded up to order 30.

Since the method requires many iterations, it is important to check the numerical stability of the algorithm. One potential numerical difficulty is

the appearance of large numbers in (2.4), since one expects (see Section 3 of ref. 8) that for large n ,

$$\left\langle \left(\sum_x \sigma_x \right)^{2m} \right\rangle_n \propto 2^{nm} \tag{2.5}$$

This difficulty can be overcome either by replacing (2.1) by a recursion formula for the logarithm of \hat{P}_n divided by the number of sites, or by introducing an appropriate rescaling in (2.1). In the following, we only discuss the magnetic susceptibility and consequently the second possibility is the most convenient one. In general, a rescaled version of (2.1) can be obtained by defining

$$R_{\lambda,n}(k) = \hat{P}_n \left(\frac{k}{\lambda^n} \right) \tag{2.6}$$

In terms of this rescaled quantity, (2.1) becomes

$$R_{\lambda,n+1}(k) = C_{n+1} \exp \left[-\frac{1}{2} \beta \left(\frac{c}{4} \lambda^2 \right)^{n+1} \frac{\partial^2}{\partial k^2} \right] \left[R_{\lambda,n} \left(\frac{k}{\lambda} \right) \right]^2 \tag{2.7}$$

The large-volume behavior shown in (2.5) can then be compensated by choosing $\lambda = \sqrt{2}$. This choice has been used for all the numerical results given hereafter. This modification improves significantly the precision of the calculation as discussed below. Note also that the renormalization group transformation is obtained by taking $\lambda = 2c^{-1/2}$ in (2.7).

In the rest of this section and the next section, we discuss mostly the numerical results for $D = 3$. Results for other values D will be discussed in Sections 4 and 5. Checking the numerical results obtained with the rescaled recursion formula (2.7) at $\lambda = \sqrt{2}$ for the first four coefficients for all n up to 60 with the exact results at finite volume,⁽⁷⁾ we found errors of at most 1 in the 13th significant digit. This precision is almost twice better than the one obtained with the original formula (2.1). We have also compared slightly different numerical implementations for which the roundoff errors should be different and found agreement at that level of precision for the 30 coefficients. We conclude that after 60 iterations, the numerical errors are at most in the 13th significant digit.

The next question is to determine how close the coefficients calculated with 2^{60} sites are to the infinite-volume ones. In order to get a first idea concerning the volume dependence of the coefficient, the results for $D = 3$ and $n = 10, 20, 30$, and 40 are given in Table I. It is possible to compare

Table I. Coefficients of High-Temperature Expansion of Susceptibility up to Order 20 for Hierarchical Model with $D=3$ and 2^n Sites with $n = 10, 20, 30,$ and 40^a

| m | $b_{m,n}$ hierarchical model, $D = 3$ | | | | $b_m^{sc}/(b_1^{HM}/b_1^{sc})^m$ simple cubic |
|-----|---------------------------------------|-----------|-----------|-----------|--|
| | $n = 10$ | $n = 20$ | $n = 30$ | $n = 40$ | |
| 1 | 1.2258497 | 1.2424374 | 1.2426008 | 1.2426024 | 1.2426024 |
| 2 | 1.2389539 | 1.2798863 | 1.2802922 | 1.2802962 | 1.2867173 |
| 3 | 1.1820851 | 1.2490897 | 1.2497599 | 1.2497665 | 1.3146331 |
| 4 | 1.0785253 | 1.1703159 | 1.1712435 | 1.1712527 | 1.2987587 |
| 5 | 0.9628730 | 1.0760627 | 1.0772198 | 1.0772312 | 1.2804051 |
| 6 | 0.8452212 | 0.9755497 | 0.9768990 | 0.9769123 | 1.2429628 |
| 7 | 0.7316955 | 0.8746523 | 0.8761526 | 0.8761674 | 1.2051755 |
| 8 | 0.6274001 | 0.7785983 | 0.7802084 | 0.7802243 | 1.1594697 |
| 9 | 0.5339023 | 0.6893634 | 0.6910447 | 0.6910612 | 1.1146199 |
| 10 | 0.4510918 | 0.6073781 | 0.6090960 | 0.6091130 | 1.0663937 |
| 11 | 0.3787313 | 0.5329771 | 0.5347017 | 0.5347187 | 1.0196561 |
| 12 | 0.3163770 | 0.4662812 | 0.4679872 | 0.4680040 | 0.9718193 |
| 13 | 0.2631594 | 0.4069559 | 0.4086227 | 0.4086391 | 0.9258158 |
| 14 | 0.2179946 | 0.3543998 | 0.3560112 | 0.3560270 | 0.8799407 |
| 15 | 0.1798494 | 0.3079961 | 0.3095398 | 0.3095551 | 0.8360504 |
| 16 | 0.1478169 | 0.2671874 | 0.2686545 | 0.2686690 | 0.7929596 |
| 17 | 0.1210802 | 0.2314434 | 0.2328281 | 0.2328417 | 0.7518815 |
| 18 | 0.0988810 | 0.2002368 | 0.2015355 | 0.2015483 | |
| 19 | 0.0805232 | 0.1730514 | 0.1742628 | 0.1742747 | |
| 20 | 0.0653886 | 0.1494050 | 0.1505292 | 0.1505403 | |

^aThe last column is for the nearest neighbor model on a simple cubic lattice.⁽¹⁰⁾ The temperature of the nearest neighbor model has been rescaled in such way that the first coefficients of the last two columns coincide.

the values of the first four coefficients with their infinite-volume-limit values⁽⁷⁾:

$$\begin{aligned}
 b_1 &= 1.242602432206561 \\
 b_2 &= 1.280296258947416 \\
 b_3 &= 1.249766644307651 \\
 b_4 &= 1.171252787942138
 \end{aligned}
 \tag{2.8}$$

One sees that, roughly, the agreement with the infinite-volume limit improves by two digits each time that n is increased by 10. This suggests that the volume dependence is exponentially suppressed. In order to check this possibility, we plot in Fig. 1 the logarithm of $b_{m,60} - b_{m,n}$ for $n < 60$

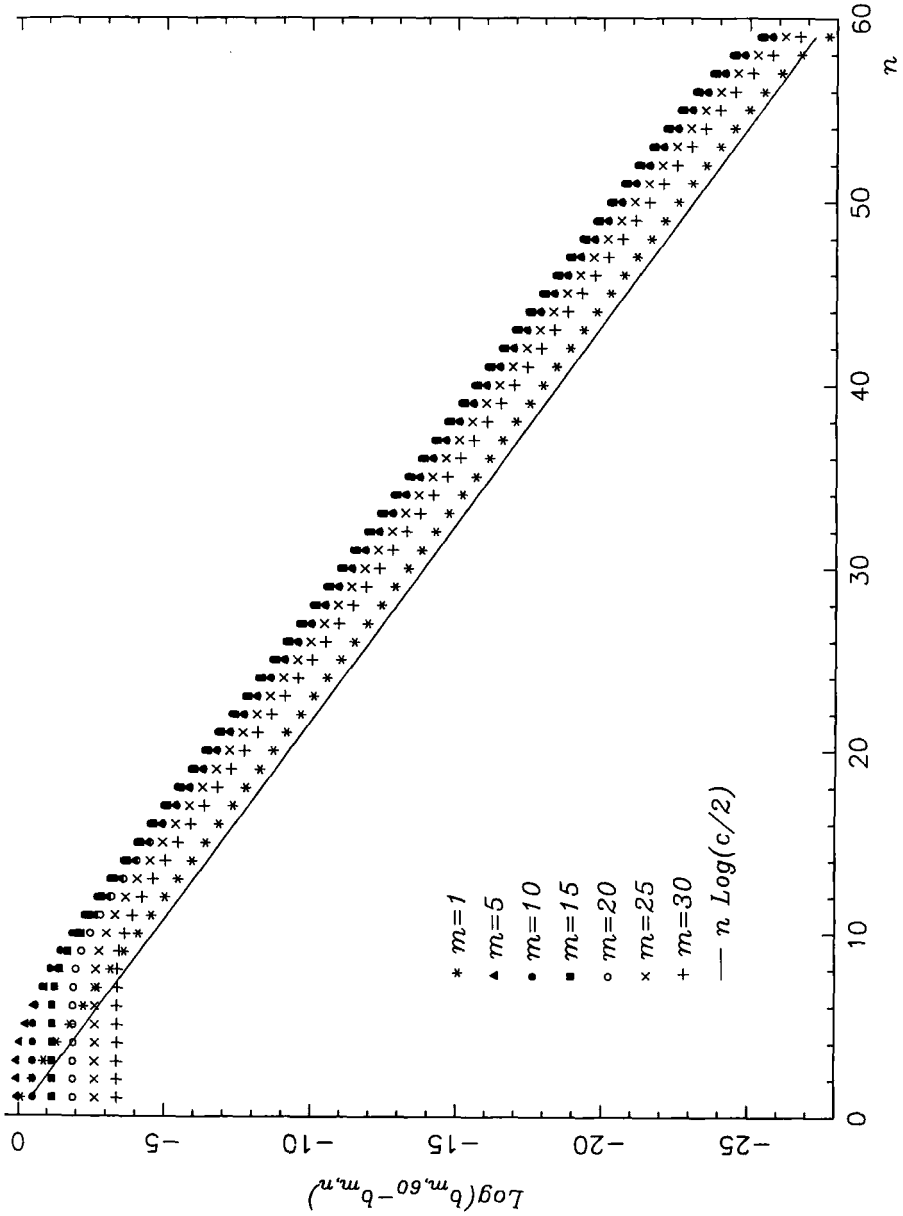


Fig. 1. The logarithm of the difference between the largest volume value $b_{m,60}$ and $b_{m,n}$ for $n < 60$, plotted for $m = 1, 5, 10, 15, 20, 25$, and 30 . The solid line is the analytical slope for low m . The value of D is 3.

and $m = 1, 5, 10, 15, 20, 25,$ and 30 . We selected these few values of m in order to keep the distinction among various values of m visible on the figure. The trajectories for other m fall approximately within the envelope delimited by the ones displayed. The figure makes clear that the length of the plateau at low n increases with m , in other words, higher order coefficients take a larger volume to stabilize (with exponential precision) near their infinite-volume limit. After this, all the trajectories appear more or less parallel. The slope can be compared with the analytical results,⁽⁷⁾ which show that the leading n -dependence of the first four coefficients comes from terms proportional to $(c/2)^n$. We recall that in the case considered here

Table II. Coefficients of High-Temperature Expansion of Susceptibility up to Order 30 for Hierarchical Model with $D=3$ and 4 with 2^{60} Sites

| m | $D=3$ | $D=4$ |
|-----|--------------|--------------|
| 1 | 1.2426024322 | 1.8672953994 |
| 2 | 1.2802962589 | 3.0879661428 |
| 3 | 1.2497666443 | 4.9423326203 |
| 4 | 1.1712527879 | 7.7505802450 |
| 5 | 1.0772313925 | 12.036610824 |
| 6 | 0.9769124559 | 18.565938390 |
| 7 | 0.8761675869 | 28.502628210 |
| 8 | 0.7802244571 | 43.615416069 |
| 9 | 0.6910614366 | 66.575570826 |
| 10 | 0.6091131879 | 101.42207830 |
| 11 | 0.5347188783 | 154.26742310 |
| 12 | 0.4680042277 | 234.35472026 |
| 13 | 0.4086393523 | 355.65012974 |
| 14 | 0.3560272432 | 539.25066450 |
| 15 | 0.3095552571 | 817.02121529 |
| 16 | 0.2686691868 | 1237.0802615 |
| 17 | 0.2328419183 | 1872.0705677 |
| 18 | 0.2015484692 | 2831.6298782 |
| 19 | 0.1742748982 | 4281.1957614 |
| 20 | 0.1505404954 | 6470.3600007 |
| 21 | 0.1299129156 | 9775.6156412 |
| 22 | 0.1120106353 | 14764.791847 |
| 23 | 0.0964971984 | 22294.166403 |
| 24 | 0.0830737698 | 33654.812081 |
| 25 | 0.0714737313 | 50793.117545 |
| 26 | 0.0614596993 | 76643.050521 |
| 27 | 0.0528217428 | 115626.74582 |
| 28 | 0.0453756513 | 174408.63341 |
| 29 | 0.0389607903 | 263031.43468 |
| 30 | 0.0334376111 | 396627.26868 |

($D = 3$), $c = 2^{1/3}$. The general agreement with this prediction is quite clear in Fig. 1. In conclusion, Fig. 1 supports convincingly the possibility that the coefficients displayed reach their infinite-volume limit with an exponential precision, and that up to the 30th coefficient for $D = 3$, the $n = 60$ results give estimates of the infinite-volume values which are accurate up to the 12th significant digits (which is the limit of our numerical precision). The numerical results are given in Table II for $n = 60$ and $D = 3$ and also, for further reference, for $D = 4$.

For comparison, we also give in Table I the first 17 coefficients for the nearest neighbor model in three dimensions on a simple cubic lattice. These coefficients have been calculated from the tables given in ref. 10. Note that these tables are given for the expansion parameter $\tanh(\beta)$, and an additional expansion is necessary in order to obtain an expansion of the form of Eq. (2.5). In addition, we rescaled the temperature of the nearest neighbor model in such a way that its first coefficient coincides with the one of the hierarchical model in the infinite-volume limit. The large discrepancies indicate that the high-temperature coefficients should be sensitive to "nonhierarchical" perturbations.

Before closing this section, let us mention the possibility of using Eq. (2.7) to obtain analytical results. This equation relates the average value of the $2m$ th power of the mean spin expanded to a certain order in β to average values of larger m and lower order in β . This suggests a bootstrap procedure which is presently under investigation. Note also that the $(c/2)^n$ rule for the volume dependence appearing in Fig. 1 can be inferred from Eq. (2.7) with $\lambda = \sqrt{2}$ with (2.5). Finally, let us mention that when n goes to infinity, the recursion formula has a fixed point of the form $\exp(-\frac{1}{2}\chi k^2)$. The linearized transformation about this fixed point is presently under study.

3. APPROXIMATE RELATIONS AMONG THE HIGH-TEMPERATURE COEFFICIENTS

In a recent numerical calculation,⁽⁸⁾ we found that the numerical data for the susceptibility at $n = 14, 15$, and 16 can be fitted very precisely with a simple power law of the form $(1 - \beta/\beta_0)^{-g}$ in the *whole* high-temperature region. As a consequence, it is possible to obtain a simple approximate formula for the high-temperature coefficients of the susceptibility in terms of g and β_0 by comparing Eq. (2.5) and

$$\left(1 - \frac{\beta}{\beta_0}\right)^{-g} = 1 + \frac{g}{\beta_0} + \frac{g(g+1)}{2\beta_0^2} + \dots \quad (3.1)$$

Approximate values for g and β_0 are obtained by comparing the two first terms of Eqs. (3.1) and (2.5). Solving $g/\beta_0 = b_{1,n}$ together with $g(g+1)/2\beta_0^2 = b_{2,n}$, we obtain

$$\begin{aligned} g &= \left(\frac{2b_{2,n}}{b_{1,n}^2} - 1 \right)^{-1} \\ \beta_0 &= \left(\frac{2b_{2,n}}{b_{1,n}} - b_{1,n} \right)^{-1} \end{aligned} \quad (3.2)$$

By comparing Eq. (2.5) with Eq. (3.1) and plugging in the values of β_0 and g given by Eq. (3.2), we obtain approximate values of the coefficients denoted $b_{m,n}^{\text{appr}}$. As an example, at third order we obtain

$$b_{3,n}^{\text{appr}} = \frac{g(g+1)(g+2)}{6\beta_0^3} \quad (3.3)$$

Using Eq. (3.2), we then obtain

$$b_{3,n}^{\text{appr}} = \frac{b_{2,n}}{3} \left(\frac{4b_{2,n}}{b_{1,n}} - b_{1,n} \right) \quad (3.4)$$

Note that the approximate relations are homogeneous in the sense that at order m , all the terms have the form $b_{1,n}^q b_{2,n}^p$ with $2p+q=m$.

In order to see how well these approximate relations are satisfied, we have calculated the difference between the exact and the approximate values of the coefficients, denoted

$$\Delta_{m,n} = b_{m,n} - b_{m,n}^{\text{appr}} \quad (3.5)$$

Note that due to the homogeneity of the approximate relations mentioned above, the ratio $\Delta_{m,n}/b_{m,n}$ is invariant under a rescaling of the temperature. In particular, this implies that the rescaling of β is used in Table I for the nearest neighbor model has no effect on this ratio. The numerical values of this ratio for $D=3$, m up to 30, and $n=40$ and 60 are displayed in Table III. The comparison between the results for the two values of n shows that the volume dependence is not an issue for the discussion in the rest of this section.

We can now compare the approximate relations for the hierarchical model and the nearest neighbor model. We see that in both cases the quality of the approximation deteriorates when m increases. However, the rates are quite different: in the nearest neighbor case, $\Delta_{m,n}/b_{m,n}$ increases by approximately 0.04 each time m is increased by 1, while the corresponding increase is approximately 0.02 (and decreases with m) for the hierarchical model.

Table III. Difference Between Exact and Approximate Values of First 30 Coefficients Divided by Exact Value, for Hierarchical Model with $D = 3$ and 2^n Sites with $n = 40$ and 60^a .

| m | $\Delta_{m,n}/b_{m,n}$ hierarchical model, $D = 3$ | | $\Delta_{m,n}^{sc}/b_{m,n}^{sc}$ simple cubic |
|-----|--|--------------|--|
| | $n = 40$ | $n = 60$ | |
| 3 | 0.0169805368 | 0.0169805378 | 0.0540540 |
| 4 | 0.0305918123 | 0.0305918165 | 0.1076487 |
| 5 | 0.0482486380 | 0.0482486469 | 0.1751963 |
| 6 | 0.0671925761 | 0.0671925914 | 0.2374947 |
| 7 | 0.0860796469 | 0.0860796701 | 0.3020008 |
| 8 | 0.1059520949 | 0.1059521270 | 0.3614195 |
| 9 | 0.1266286057 | 0.1266286476 | 0.4191415 |
| 10 | 0.1473343900 | 0.1473344426 | 0.4719078 |
| 11 | 0.1679293974 | 0.1679294615 | 0.5216787 |
| 12 | 0.1886442528 | 0.1886443288 | 0.5669304 |
| 13 | 0.2094865664 | 0.2094866545 | 0.6089341 |
| 14 | 0.2302279746 | 0.2302280751 | 0.6469778 |
| 15 | 0.2506805821 | 0.2506806952 | 0.6819434 |
| 16 | 0.2708136887 | 0.2708138145 | 0.7135224 |
| 17 | 0.2906785507 | 0.2906786892 | 0.7423546 |
| 18 | 0.3103045616 | 0.3103047126 | |
| 19 | 0.3296623492 | 0.3296625126 | |
| 20 | 0.3486881339 | 0.3486883093 | |
| 21 | 0.3673252136 | 0.3673254009 | |
| 22 | 0.3855482352 | 0.3855484341 | |
| 23 | 0.4033616171 | 0.4033618273 | |
| 24 | 0.4207829386 | 0.4207831598 | |
| 25 | 0.4378265287 | 0.4378267605 | |
| 26 | 0.4544958062 | 0.4544960481 | |
| 27 | 0.4707843981 | 0.4707846497 | |
| 28 | 0.4866817079 | 0.4866819688 | |
| 29 | 0.5021786614 | 0.5021789312 | |
| 30 | 0.5172713168 | 0.5172715950 | |

^aThe last column is the same quantity for the nearest neighbor model on a simple cubic lattice.

Equation (3.2) is exactly what we would have obtained from a $[0, 1]$ Padé approximant for the logarithmic derivate of the susceptibility. In the infinite-volume limit, we obtain $g = 1.5189$ and $\beta_0 = 1.2224$. The value of g is significantly larger than the value of the critical exponent γ obtained⁽⁹⁾ in the ϵ -expansion, namely 1.300. A value close to 1.30 has also been obtained with independent methods by Bleher and Sinai⁽³⁾. The value of β_0 is also significantly larger than our best estimate⁽⁸⁾ $\beta_c = 1.179$. It is not a surprise that a $[0, 1]$ Padé approximant provides bad estimates of the

critical quantities. What is a surprise is how slowly the high-temperature series of the hierarchical model departs from this poor approximation when the order m increases. This slower rate of departure implies that a number of coefficients much larger than in the nearest neighbor case is necessary in order to estimate the critical quantities with a comparable precision.

We suspect that there exists a definite relationship between the quality of the estimates made from finite n and infinite m (as done in ref. 8) and those made from finite m and infinite (or sufficiently large) n (as done here). This remark is motivated by the observation that the values of γ obtained at $n = 16$ (e.g., 1.47 for $D = 3$) are close to those obtained from low-order Padé approximants (see next section). The discrepancy of the finite-volume fits with the ε -expansion was attributed⁽⁸⁾ to the fact that for $n = 16$, for instance, we can only use the recursion formula 16 times, while a larger number seems necessary to get rid of the irrelevant components of the measure. This statement has a counterpart for the high-temperature expansion, which is roughly the following: the coefficients which at a finite n have reached their infinite value with an acceptable precision provide estimates of the critical exponents consistent with the finite- n data. Since a direct calculation of the susceptibility at $n = 30$ or 40 seems excluded with the exact method used in ref. 8, we cannot sharpen further the statement made above. However, the n dependence of the numerical estimates shown in Figs. 11 and 12 of ref. 8, the regularity of the length of the shoulders of Fig. 1 of the present paper, and the results shown below suggest that a more quantitative study could be made in the future, possibly using Eq. (2.7).

4. ESTIMATION OF β_c AND γ USING PADÉ APPROXIMANTS

In this section, we discuss the estimation of β_c and γ , using $[L, M]$ Padé approximants with $L + M + 1 \leq 30$. In the following, we call $L + M + 1$ the *order* of the approximant. Since our calculation of the coefficients has been done at large but finite volume, we will first address the question of the volume dependence of the results obtained. As well known, at finite volume, a singularity of the susceptibility on the real temperature axis is impossible. However, when the volume increases, a pair of complex conjugate singularities can approach arbitrarily closely the real axis. As we now proceed to explain, the Padé approximants sense this approach of the real axis, but with a finite resolution.

We studied a large set of approximants using the finite-volume coefficients and followed the motion of the roots of the denominator when the volume is increased. We observed the following patterns in a large majority of cases. For n below a number which is usually between 10 and 15, the

roots move by large steps, usually of the same order as the change in the coefficients. During this preliminary motion, a pair of roots “pinches” the real axis and two clearly distinct real roots appear. This situation is illustrated in Fig. 2. One of these two real roots almost immediately stabilizes near β_c (1.179 in this case), while the other makes a few erratic steps before stabilizing (more slowly) near another real value clearly distinct from β_c . Figure 3 illustrates the small size motion of all the roots in the complex plane for n between 17 and 60. All the roots are well separated and for the roots labeled 1, 3, 4, 5, and 6 the changes are barely perceptible. We skipped $n = 14, 15,$ and $16,$ because the root labeled as 2 in Fig. 2 makes sudden changes during these three steps. Comparing with other cases, it appeared clearly that the value of n for which the pinching of the real axis occurs increases with the order $(L + M + 1)$ of the approximant. In other words, it appears that the Padé approximants of a given order can only resolve a pair of complex conjugate roots when their distance to the real axis is not less than a certain value (which decreases with the order of the approximant).

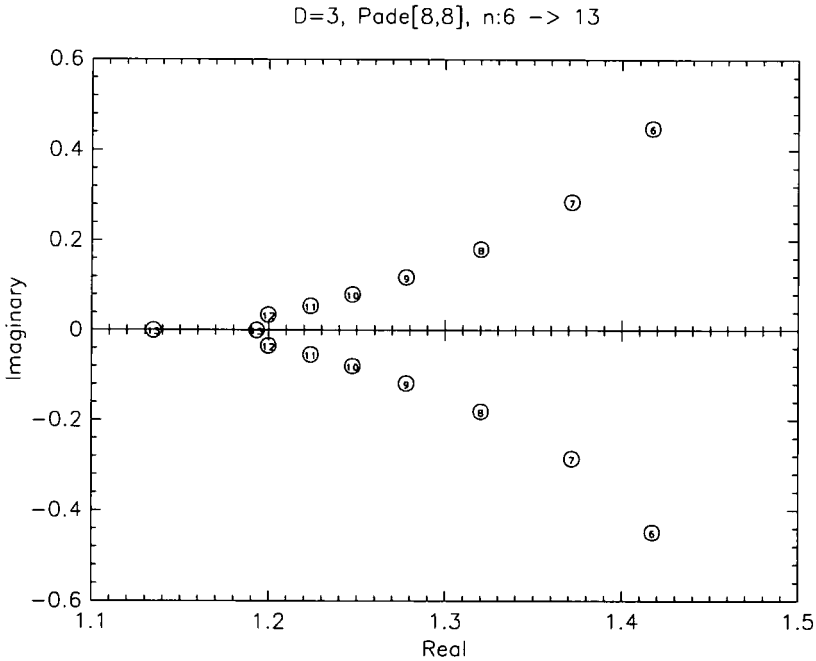


Fig. 2. The motion of the pair of complex conjugate roots near the critical value for n between 6 and 13. A [8, 8] Padé approximant was used for $D = 3$.

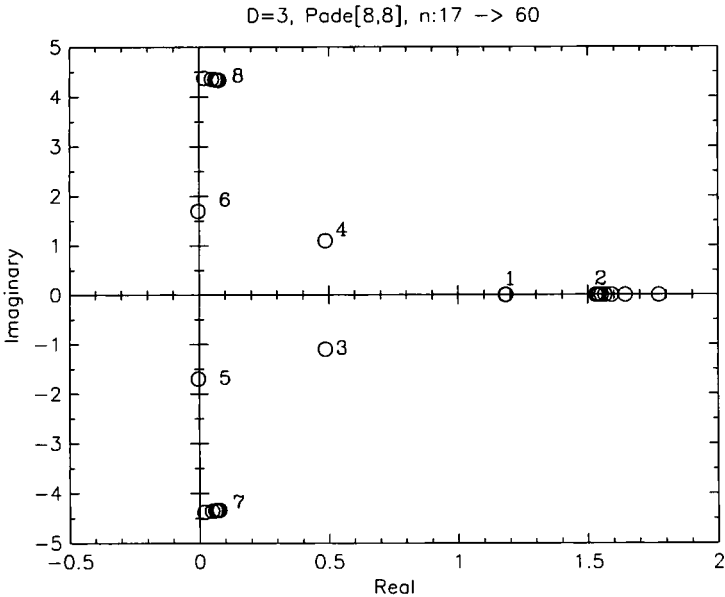


Fig. 3. The motion of all the roots for n between 17 and 60 with a [8, 8] Padé approximant and $D=3$.

In most of the cases, the roots are widely separated for $n > 20$. However, there are a few exceptional cases where two real roots are very close to β_c , for instance, the [13, 12] and the [12, 13] approximants for $D=3$, and also [10, 11] and [11, 12] for $D=4$. In these special cases, the residues at both poles are very different from the residues at the pole near β_c for most approximants of comparable order.

A closer look at the numbers indicates that the change in the location of the pole near 1.18 and the change (due to the volume) in the value of the residue at the pole decrease exponentially when the volume is increased. In the case of approximants requiring four coefficients or less, we can indeed obtain a precise comparison between the results at $n=60$ and $n=\infty$ using the infinite-volume limit values of Eq. (2.8). We found that for the six approximants with $L+M+1 \leq 4$ the difference between the $n=60$ and $n=\infty$ appears at worst in the 11th significant digit for both β_c and γ . We also made independent random changes of order 10^{-4} in the values of a few coefficients and did not observe any instability in the estimations. Our general conclusion concerning the volume dependence of the estimations obtained from Padé approximants calculated with coefficients corresponding to models with 2^{60} sites is that the results obtained approximate

Table IV. Values of β_c and $-\gamma$ (in parentheses) for $D=3$ Obtained from $[j+k, j]$ Padé Approximants, for $1 \leq j \leq 14$ and $k = \pm 1$ and 0^a

| j | $\beta_c (-\gamma)$ | | |
|-----|----------------------|----------------------|----------------------|
| | $k = -1$ | $k = 0$ | $k = 1$ |
| 1 | 1.222394 (-1.518949) | 1.135464 (-1.310594) | 1.246632 (-1.734454) |
| 2 | 1.183125 (-1.449299) | 1.195448 (-1.494930) | 1.181455 (-1.421848) |
| 3 | 1.189636 (-1.469088) | 1.187859 (-1.459889) | 1.187994 (-1.460775) |
| 4 | 1.187982 (-1.460682) | 1.187864 (-1.459908) | 1.183228 (-1.423736) |
| 5 | 1.189636 (-1.467927) | 1.183937 (-1.430664) | 1.183605 (-1.427262) |
| 6 | 1.183235 (-1.422855) | 1.184452 (-1.434872) | 1.180596 (-1.388929) |
| 7 | 1.193406 (-1.397933) | 1.181859 (-1.408136) | 1.181538 (-1.403356) |
| 8 | 1.181416 (-1.401304) | 1.180724 (-1.387517) | 1.181836 (-1.407342) |
| 9 | 1.181619 (-1.404328) | 1.182650 (-1.414949) | 1.180967 (-1.394885) |
| 10 | 1.179699 (-1.366305) | 1.179873 (-1.370865) | 1.180268 (-1.380805) |
| 11 | 1.179632 (-1.364583) | 1.180326 (-1.382215) | 1.180282 (-1.381124) |
| 12 | 1.180177 (-1.378432) | 1.178694 (-1.322758) | 1.173194 (-0.92856*) |
| 13 | 1.171789 (-0.81937*) | 1.177754 (-1.271869) | 1.180257 (-1.382405) |
| 14 | 1.179944 (-1.373549) | 1.179709 (-1.365857) | 1.179725 (-1.366443) |

^a An asterisk indicates two very nearby poles.

the infinite-volume results with a precision which is far beyond what would be required to discuss effects of the order of 10^{-3} in the critical quantities.

We can now present the numerical results. In Tables IV and V we give the values of β_c and γ obtained from the $[j+k, j]$ approximants for $1 \leq j \leq 14$, $k = \pm 1, 0$, and $D=3$ and 4. In both cases, we have used the

Table V. Same Quantities as in Table IV, but for $D=4$

| j | $\beta_c (-\gamma)$ | | |
|-----|-----------------------------------|----------------------|----------------------|
| | $k = -1$ | $k = 0$ | $k = 1$ |
| 1 | 0.694383 (-1.296619) | 0.665722 (-1.191788) | 0.681561 (-1.278896) |
| 2 | 0.675868 (-1.240424) | 0.674069 (-1.229981) | 0.670338 (-1.200979) |
| 3 | 0.680365 (-1.241321) | 0.670043 (-1.197792) | 0.670399 (-1.201272) |
| 4 | 0.668001 (-1.170086) | 0.663631 (-1.071941) | 0.667729 (-1.168325) |
| 5 | 0.666688 (-1.147293) | 0.667201 (-1.157996) | 0.663790 (-1.037130) |
| 6 | 0.666958 (-1.152793) | 0.666199 (-1.132816) | 0.666550 (-1.143434) |
| 7 | 0.666418 (-1.139410) | 0.666418 (-1.139406) | 0.665664 (-1.108706) |
| 8 | 0.666418 (-1.139410) | 0.666009 (-1.124682) | 0.666005 (-1.124495) |
| 9 | 0.666005 ² (-1.124491) | 0.666009 (-1.124668) | 0.666034 (-1.125764) |
| 10 | 0.666030 (-1.125610) | 0.666018 (-1.125068) | 0.665965 (-1.122604) |
| 11 | 0.664996 (-1.03404*) | 0.665567 (-1.095758) | 0.665791 (-1.112854) |
| 12 | 0.668440 (-1.00079*) | 0.665818 (-1.114617) | 0.665807 (-1.113874) |
| 13 | 0.665800 (-1.113384) | 0.666001 (-1.122615) | 0.665969 (-1.121304) |
| 14 | 0.665937 (-1.120872) | 0.667435 (-1.129101) | 0.667029 (-1.169627) |

coefficients for $n=60$. We can now compare these results with our best estimates. In the case $D=3$, one observes a very slow (compared to nearest neighbor models) progression toward $\beta_c = 1.179$ and^(3,9) $\gamma = 1.300$ when the order of the approximants is increased. In the case $D=4$, there is a similar progression toward $\beta_c = 0.665$ and the trivial value $\gamma = 1$.

In order to give a more complete idea concerning the results obtained from Padé approximants, we give in Fig. 4 the distribution of values γ obtained from the 435 different approximants of order less than 30 for $D=3$. Less than 0.5% of the data fell outside of the figure. The figure shows separately the distributions for $L+M \leq 10$, $10 < L+M \leq 20$, and $20 < L+M < 30$. The figure makes clear that the distribution gets more peaked and that the average decreases when the order of the approximants increases. The average for the distribution with $20 < L+M < 30$ is 1.382, which is still far from 1.300. Note also that the number of approximants giving a value of γ near 1.30 is nonnegligible for $L+M < 10$; however, this feature clearly fades away when the order increases.

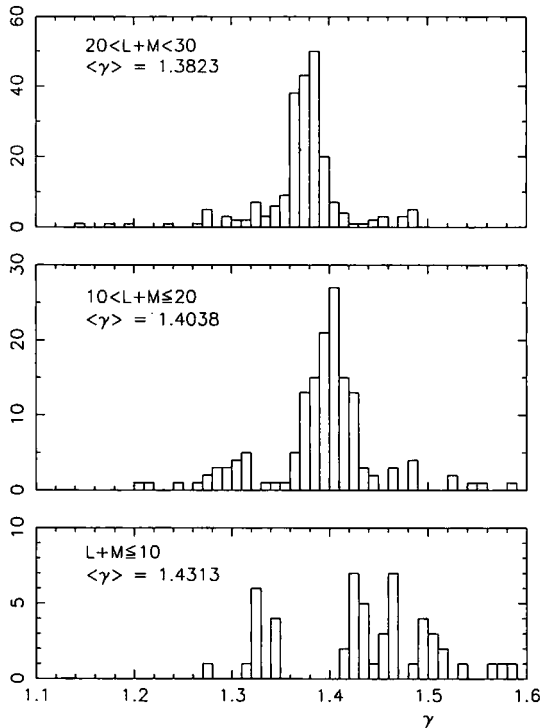


Fig. 4. Distribution of the values of γ obtained from Padé approximants with $L+M+1 \leq 30$ and $D=3$.

5. ARE THERE MODELS FOR WHICH THE APPROXIMATE RELATIONS ARE EXACT?

Up to now, the approximate relations discussed in Section 3 have had a rather unpleasant effect: we need to calculate many coefficients in order to get any reliable result. However, the situation would be opposite if these relations were exact or if the departure could be estimated very precisely. It is thus tempting to try to construct models where, for instance, the $[0, 1]$ Padé approximant would be exact.

A modest step in this direction consists in considering the D dependence of the approximate relations studied in the special case $D = 3$ in Section 3 and comparing with the nearest neighbor models. In the rest of this section, we consider the infinite-volume limit and the reference to n disappears. In Fig. 5 we plot Δ_m/b_m for $m = 3$ and 4 as a function of D using the exact analytical results of ref.7. We see that both quantities decrease when D becomes large and vanish for special values of D between 1 and 2. A more detailed graph would show that Δ_3 vanishes near $D = 1.14$ and Δ_4 near $D = 1.66$. It is not clear that we can learn anything interesting from the behavior of the coefficients in this low- D region. On the other hand, the fact that the approximation becomes better for large D is due to the fact that, as for the nearest neighbor models,⁽¹²⁾ the corrections to the Gaussian result $b_m/b_1^m = 1$ vanish. Note that the approximation discussed

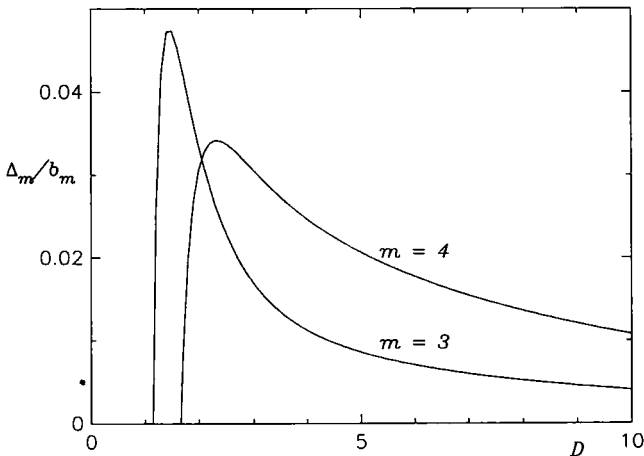


Fig. 5. The difference between the exact and approximate values of the third and fourth coefficients divided by the exact value, as a function of D . The analytical results of ref. 8 in the infinite-volume limit were used to draw the curves.

in Section 3 is more general than the Gaussian approximation, since it allows b_2 to be distinct from b_1^2 ; however, it is obviously exact in the Gaussian approximation.

We first consider the nearest neighbor case. After an elementary calculation and an appropriate expansion of $\tanh(\beta)$, we obtain that for the Ising models with nearest neighbor interactions on a hypercubic lattice of dimension D : $b_1 = 2D$, $b_2 = -2D + 4D^2$, $b_3 = 4D/3 - 8D^3$, $b_4 = 10D/3 + 16D^2/3 - 24D^3 + 16D^4$.

Expanding Δ_m/b_m in $1/D$, we obtain

$$\left(\frac{\Delta_3}{b_3}\right) = \frac{1}{6D} - \frac{1}{36D^3} - \frac{1}{36D^4} - \frac{5}{216D^5} + O\left(\frac{1}{D}\right)^6 \quad (5.1)$$

and

$$\left(\frac{\Delta_4}{b_4}\right) = \frac{5}{12D} - \frac{1}{4D^2} - \frac{1}{18D^3} - \frac{25}{288D^4} - \frac{103}{1728D^5} + O\left(\frac{1}{D}\right)^6 \quad (5.2)$$

We can now compare with a similar expansion for the hierarchical model. Using the results of Section 6 of ref. 7, we obtain

$$\begin{aligned} \left(\frac{\Delta_3}{b_3}\right) &= \frac{8 \log(2)^2}{3D^2} - \frac{512 \log(2)^3}{9D^3} \\ &+ \frac{2488 \log(2)^4}{3D^4} - \frac{266432 \log(2)^5}{27D^5} + O\left(\frac{1}{D}\right)^6 \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \left(\frac{\Delta_4}{b_4}\right) &= \frac{20 \log(2)^2}{3D^2} - \frac{1096 \log(2)^3}{9D^3} \\ &+ \frac{80084 \log(2)^4}{63D^4} - \frac{2764360 \log(2)^5}{441D^5} + O\left(\frac{1}{D}\right)^6 \end{aligned} \quad (5.4)$$

It is clear that the corrections to the Gaussian result appear at a higher order $[(1/D)^2]$ for the hierarchical model than for the nearest neighbor model. However, a direct comparison of the two expansions does not shed any new light on Table III. The reason is that for $D = 3$, the series given in Eqs. (5.3)–(5.4) diverge. The origin of these divergences can be found from the zeros and poles of the coefficients in the complex $1/D$ plane (see ref. 7 for more detail). For instance, in the case of Eq. (5.3), we found a pair of poles at $(1/D) = -0.15608 \pm i0.042037$ and then at $(1/D) = 0.25$, etc. A complete description seems to be of little interest, since it does not suggest any way to improve the approximation.

We end this section with a speculative note. Supersymmetric gauge theories provide nontrivial examples where the dependence of some Green's functions on the mass is expected to be a simple power law,⁽¹³⁾ which is somehow the same thing as saying that Eq. (3.1) is exact. It is clearly difficult to compare these supersymmetric theories, which as far as we know do not have a satisfactory lattice regularization, with the spin models considered here. However, the nonlocality of the hierarchical model suggests that it is an "effective" theory obtained after integrating over some other local variables. Indeed, a concrete example has been given in ref. 15, where the hierarchical model was reformulated as a local theory with additional spin variables integrated with a Gaussian measure. It thus conceivable that one could find a reformulation of the hierarchical model where approximate symmetries playing a role similar to the supersymmetries could be more manifest.

6. CONCLUSIONS

We have presented a recursive method of calculation which allows an efficient calculation of the high-temperature expansion of the hierarchical Ising model. The number of coefficients calculated is larger than what seems achievable^(10, 11) in the case of nearest neighbor models. However, our errors on the critical exponent γ range between 0.05 and 0.1 and are much larger than the typical precision (0.001) obtained for nearest neighbor models with less than 20 coefficients. We have carefully considered the errors due to the fact that we used a large but finite volume to calculate the coefficients and concluded that these errors were much smaller than the effects described above.

We have identified the reason for this slow approach of the asymptotic regime by comparing the departure from the approximation corresponding to a $[0, 1]$ Padé approximant. The last two columns of Table III indicate that at least 50 coefficients will be necessary in order to get a precision of 0.001 for γ . However, Tables IV and V seem to leave open the possibility that a much larger number might be necessary. With the numerical method used in this paper, it would probably take a year to calculate 400 coefficients. We are presently working on an optimized version⁽¹⁶⁾ of the existing program which would allow us to achieve this goal in a shorter time.

Note that recursive methods have been found⁽¹¹⁾ to calculate the high-temperature expansion of the 3D Ising model with nearest neighbor interactions. Their recursive step consists in adding one spin and the calculation requires large parallel computers to calculate 24 coefficients. Our recursive step consists in putting in contact two identical systems of size 2^n in order to get a system of size 2^{n+1} and 24 coefficients can be obtained within less

than 2 days with a PC. On the other hand, with 24 coefficients, we obtain estimates which are less precise by one order of magnitude. We would like to check whether or not the following possibility occurs: our recursive algorithm would require a logarithmically smaller time of computation, but at the same time require an exponentially larger number of coefficients in order to reach a given precision.

Having at hand a large number of coefficients for the high-temperature expansion would allow precise comparison with the ε -expansion, for which a larger number of coefficients is available.⁽⁹⁾ In the case of the nearest neighbor models, discrepancies have been observed between these two methods (for references to the extensive literature, see ref. 17). However, several authors⁽¹⁸⁾ have shown that these discrepancies can be removed by an appropriate treatment of the confluent singularities. For the hierarchical model, this is a completely open question. It is also possible that the simplicity of Eqs. (2.6)–(2.8) would allow us to obtain analytical results concerning the asymptotic behavior of the high-temperature expansion.

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REFERENCES

1. F. Dyson, *Commun. Math. Phys.* **12**:91 (1990); G. Baker, *Phys. Rev. B* **5**:2622 (1972).
2. K. Wilson, *Phys. Rev. B* **4**:3185 (1971); K. Wilson and J. Kogut, *Phys. Rep.* **12**:75 (1974).
3. P. Bleher and Y. Sinai, *Commun. Math. Phys.* **55**:247 (1975); P. Collet and J. P. Eckmann, *Commun. Math. Phys.* **55**:67 (1977); P. Collet and J. P. Eckmann, *Lecture Notes in Physics*, No. 74 (Springer-Verlag, Berlin, 1978).
4. H. Koch and P. Wittwer, *Commun. Math. Phys.* **106**:495 (1986); **138**:537 (1991); **164**:627 (1994).
5. K. Gawedzki and A. Kupiainen, in *Les Houches 1985*, K. Osterwalder and R. Stora, eds. (North-Holland, Amsterdam), p. 185.
6. Y. Meurice, in *Proceedings of the International Europhysics Conference on High Energy Physics*, J. Carr and M. Perottet, eds. (Editions Frontières, 1994), p. 89.
7. Y. Meurice, *J. Math. Phys.* **36**:1812 (1995); University of Iowa preprint 94-14 hep-lat 9408021.
8. Y. Meurice, G. Ordaz, and V. G. J. Rodgers, *J. Stat. Phys.* **77**:607 (1994).
9. P. Collet, J.-P. Eckmann, and B. Hirsbrunner, *Phys. Lett.* **71B**:385 (1977).
10. D. S. Gaunt and A. J. Guttmann, in *Phase Transitions and Critical Phenomena*, C. Domb and M. S. Green, eds. (Academic Press, New York, 1974).
11. G. Bhanot, M. Creutz, U. Glasser, and K. Schilling, *Phys. Rev. B* **49**:12909 (1994).
12. M. Fisher and D. Gaunt, *Phys. Rev.* **133A**:224 (1964).
13. G. Veneziano, *Phys. Lett.* **B124**:357 (1983); also see D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, *Phys. Rep.* **162**:169 (1988).

14. Y. Meurice, *Phys. Lett.* **265**:377 (1991); J. L. Lucio and Y. Meurice, *Mod. Phys. Lett.* **6**:1199 (1991).
15. Y. Meurice, *Mod. Phys. Lett.* **7**:3331 (1992).
16. Y. Meurice, G. Ordaz and V. G. J. Rodgers, Work in progress.
17. M. Levy, J. C. Le Guillou, and J. Zinn-Justin, eds., *Phase Transitions, Cargese 1980* (Plenum Press, New York, 1982).
18. Z. Roskies, *Phys. Rev. B* **24**:5305 (1981); J. Adler, M. Moshe, and V. Privman, *Phys. Rev. B* **26**:3958 (1982); G. Nickel and M. Dixon, *Phys. Rev. B* **26**:3965 (1982).